7.3 Volumes

Starting the same way, we can now find the volumes of a great many solids by integration. Suppose we want to find the volume of a solid like the one in Figure 7.15. The cross section of the solid at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$. If A is a continuous function of x, we can use it to define and calculate the volume of the solid as an integral in the following way.

We partition $[a, b]$ into subintervals of length Δx and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points. The kth slice, the one between the planes at x_{k-1} and x_k, has approximately the same volume as the cylinder between the two planes based on the region $R(x_k)$ (Figure 7.16).

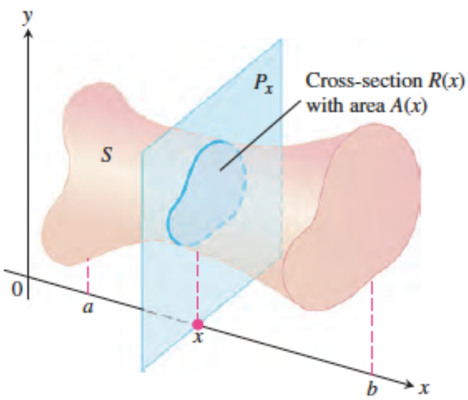
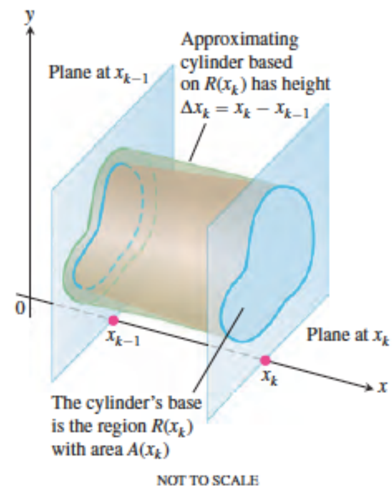


Figure 7.15 The cross section of an arbitrary solid at point x.



The volume of the cylinder is

$$V_k = \text{base area} \times \text{height} = A(x_k) \times \Delta x.$$

The sum

$$\sum V_k = \sum A(x_k) \times \Delta x$$

DEFINITION Volume of a Solid

The **volume of a solid** of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

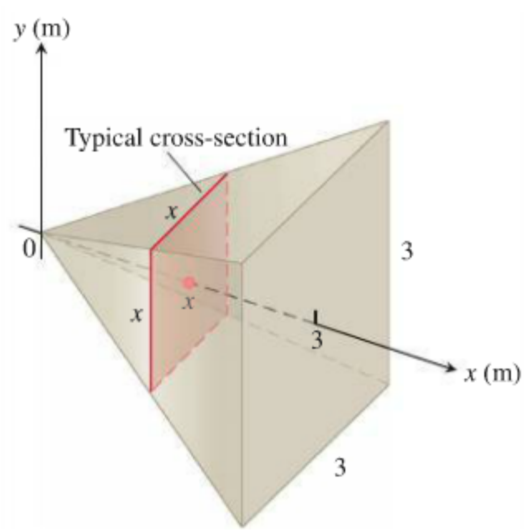
$$V = \int_a^b A(x)\, dx.$$

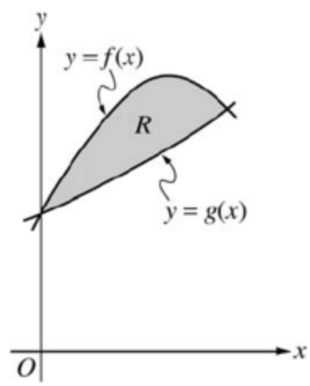
How to Find Volume by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

EXAMPLE 1 A Square-Based Pyramid

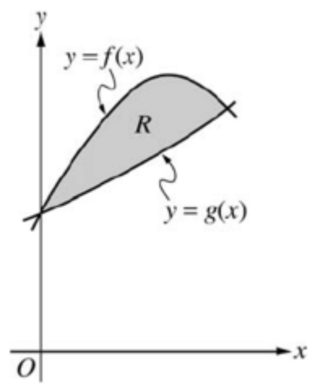
A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.





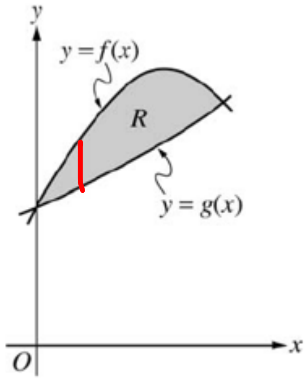
Let f and g be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let R be the shaded region in the first quadrant enclosed by the graphs of f and g as shown in the figure above.

(d) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles with diameters extending from $f(x)$ to $g(x)$. Find the volume of this solid.



Let f and g be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let R be the shaded region in the first quadrant enclosed by the graphs of f and g as shown in the figure above.

(f) The region R is the base of a solid. For this solid, the cross sections perpendicular to the *x-axis* are equilateral triangles. Find the volume of this solid.



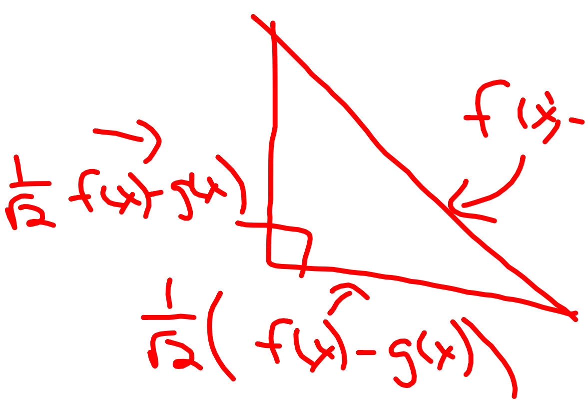
Let f and g be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let R be the shaded region in the first quadrant enclosed by the graphs of f and g as shown in the figure above.

(j) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are rectangles with height 5. Find the volume of this solid.

(k) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of this solid.

(n) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are rectangles with height twice its width. Find the volume of this solid.

h) The region R is the base of a solid. For this solid the cross sections perpendicular to the x-axis are isosceles triangles with their hypotenuse on the base. Find the volume of the solid.



$A = \frac{1}{2}bh$

$A(x) = \frac{1}{2}\cdot\frac{1}{2}(f(x)-g(x))^2$

$$\int_0^3 \frac{1}{4}(f(x)-g(x))^2\,dx$$

EXAMPLE 2 A Solid of Revolution

The region between the graph of $f(x) = 2 + x\cos x$ and the x-axis over the interval $[-2, 2]$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

$A = \pi r^2$

$r = f(x)$

$$\int_{-2}^{2} \pi f(x)^2 \, dx$$

$A(x) = \pi f(x)^2$

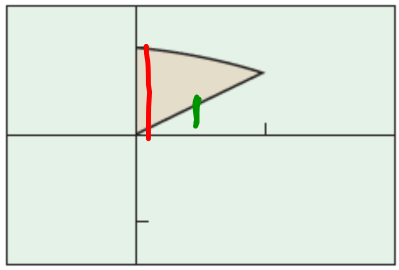
≈ 52.429

EXAMPLE 3 Washer Cross Sections

The region in the first quadrant enclosed by the y-axis and the graphs of $y = \cos x$ and $y = \sin x$ is revolved about the x-axis to form a solid. Find its volume.

$R = \cos x$

$r = \sin x$



[–π/4, π/2] by [–1.5, 1.5]

Figure 7.20 The region in Example 3.

$A_{washer} =$

$\pi R^2 - \pi r^2$

$\pi (R^2 - r^2)$

We revolve it about the x-axis to generate a solid with a cone-shaped cavity in its center (Figure 7.21).

$$\int_0^{\pi/4} \pi (\cos^2 x - \sin^2 x)\, dx$$

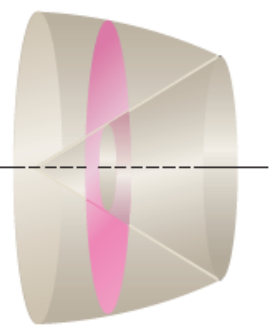


Figure 7.21 The solid generated by revolving the region in Figure 7.20 about the x-axis. A typical cross section is a washer: a circular region with a circular region cut out of its center. (Example 3)

$$\int_0^{\pi/4} \pi (\cos 2x)\, dx$$

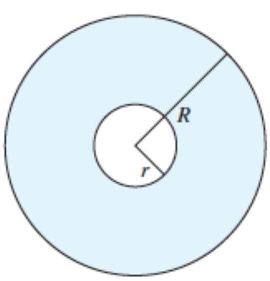
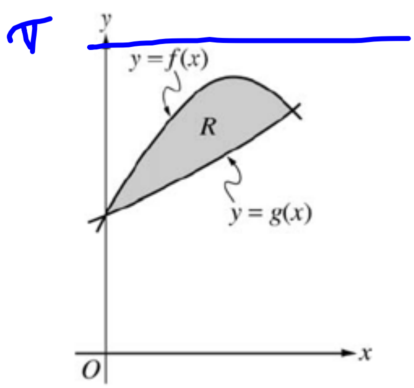


Figure 7.22 The area of a washer is $\pi R^2 - \pi r^2$. (Example 3)

$$\frac{\pi}{2} \sin 2x \Big|_0^{\frac{\pi}{4}}$$

$$\frac{\pi}{2}(1) = \frac{\pi}{2}$$



$A = \pi(R^2 - r^2)$

Let f and g be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let R be the shaded region in the first quadrant enclosed by the graphs of f and g as shown in the figure above.

(g) Find the volume of the solid generated when R is revolved around the line $y = -1$.

$R = f(x) - (-1)$

$r = g(x) - (-1)$

≈ 6.963

(m) Find the volume of the solid generated when R is revolved around the line $y = \pi$.

≈ 4.204

$R = \pi - f(x)$

$r = \pi - g(x)$

$$\pi \int_0^A \left[(\pi - y_1)^2 - ((\pi - y_2)^2\right] dx$$

EXAMPLE 7 Cavalieri's Volume Theorem

Cavalieri's volume theorem says that solids with equal altitudes and identical cross section areas at each height have the same volume (Figure 7.31). This follows immediately from the definition of volume, because the cross section area function $A(x)$ and the interval $[a, b]$ are the same for both solids.

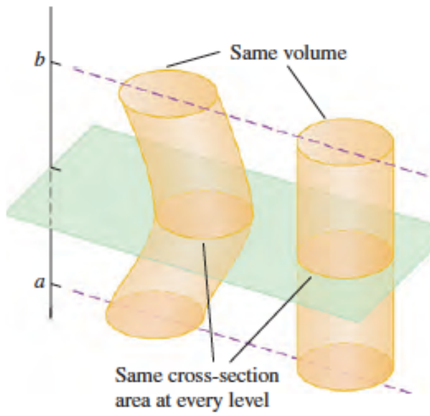


Figure 7.31 *Cavalieri's volume theorem:* These solids have the same volume. You can illustrate this yourself with stacks of coins. (Example 7)